The Schröder-Bernstein property for weakly minimal theories

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Abstract

For a countable, weakly minimal theory T, we show that the Schröder-Bernstein property (any two elementarily bi-embeddable models are isomorphic) is equivalent to each of the following:

1. For any U-rank-1 type $q \in S(\mathrm{acl}^{eq}(\emptyset))$ and any automorphism f of the monster model $\mathfrak{C}$, there is some $n < \omega$ such that $f^n(q)$ is not almost orthogonal to $q \otimes f(q) \otimes \ldots \otimes f^{n-1}(q)$;

2. T has no infinite collection of models which are pairwise elementarily bi-embeddable but pairwise nonisomorphic.

We conclude that for countable, weakly minimal theories, the Schröder-Bernstein property is absolute between transitve models of ZFC.

1 Introduction

We are concerned the following property of a first-order theory T:

Definition 1.1. A theory T has the *Schröder-Bernstein property*, or the *SB property*, if any two elementarily bi-embeddable models of T are isomorphic.

Our motivation is to find some nice model-theoretic characterization of the class of complete theories with the SB property. This property was first studied in the 1980's by Nurmagambetov in [7] and [6] (mainly within the class of ω-stable theories). In [6], he showed:

Theorem 1.2. *If T is ω-stable, then T has the SB property if and only if T is nonmultidimensional.*

One of the results from the thesis of the first author ([3]) was:

Theorem 1.3. *If a countable complete theory T has the SB property, then T is superstable, nonmultidimensional, and NOTOP, and T has no* nomadic types*; that is, there is no type $p \in S(M)$ such that there is an automorphism $f \in \mathrm{Aut}(M)$ for which the types $\{f^n(p) : n \in \mathbb{N}\}$ are pairwise orthogonal.*

In particular, any countable theory with the SB property must be classifiable (in the sense of Shelah). Within classifiable theories, the SB property seems to form a new dividing line distinct from the usual dichotomies in stability theory.

In this note, we investigate the SB property for weakly minimal theories (that is, theories in which the formula "$x = x$" is weakly minimal). We prove the following characterization, confirming a special case of a conjecture of the first author (from [3]):

Theorem 1.4. *If T is countable and weakly minimal, then the following are equivalent:*

1. T has the SB property.

2. For any U-rank-1 type $q \in S(\mathrm{acl}^{eq}(\emptyset))$ and any automorphism f of the monster model $\mathfrak{C}$, there is some $n < \omega$ such that $f^n(q)$ is not almost orthogonal to $q \otimes f(q) \otimes \ldots \otimes f^{n-1}(q)$.

3. T has no infinite collection of models which are pairwise elementarily bi-embeddable but pairwise nonisomorphic.

The proof of Theorem 1.4 uses some geometric stability theory to reduce to the case where p is the generic type of an infinite definable group, and then a Dushnik-Miller style argument can be used to construct witnesses to the failure of the SB property whenever condition 2 fails.

A corollary is that for countable, weakly minimal theories, the SB property is invariant under forcing extensions of the universe of set theory:

Corollary 1.5. *Among countable weakly minimal theories, the SB property is absolute between transitive models of ZFC containing all the ordinals.*

Proof. First, note that for a countable theory T, condition (2) of Theorem 1.4 is equivalent to the following statement:

For any *countable* model M of T, any 1-type $q \in S(M)$, and any $f \in \mathrm{Aut}(M)$, there is an $n < \omega$ such that $f^n(q)$ is not almost orthogonal to $q \otimes f(q) \otimes \ldots \otimes f^{n-1}(q)$.

Using this, it follows that among countable theories, condition (2) is a (lightface) Π^1_1 property. So the corollary follows from the Shoenfield Absoluteness Theorem for Π^1_2 relations (Theorem 98 of [5]).

□

Corollary 1.6. *If T is any countable weakly minimal theory and $T' \supseteq T$ is the expansion by new constants with one new constant naming each element of $\mathrm{acl}^{eq}_T(\emptyset)$, then T' has the SB property.*

Proof. T' trivially satisfies condition 2 of Theorem 1.4 (and is still countable and weakly minimal).

□

After discussing some preliminaries in section 2, we show in section 3 that the SB property is incompatible with the existence of a definable group with a sufficiently "generic" automorphism (Theorem 3.13). In section 4, we give another criterion for the failure of SB (Theorem 4.8) and finish with a proof of Theorem 1.4.

2 Preliminaries

We follow the usual conventions of stability theory, as explained in [8].

First, by the canonicity of Shelah's *eq*-construction, it is straightforward to see:

Fact 2.1. *If T is any theory, then T has the SB property if and only if T^{eq} does.*

For the rest of this paper, we will assume that $T = T^{eq}$ for all the theories T we consider.

A technical advantage of working with weakly minimal theories is that models are easy to construct:

Lemma 2.2. *If T is weakly minimal, $M \models T$, and $A \subseteq \mathfrak{C}$, then*

$$\operatorname{acl}(M \cup A) \models T.$$

Proof. By the Tarski-Vaught test, it is enough to check that any consistent formula $\varphi(x; \overline{b})$ over $M \cup A$ in a *single* free variable has a realization in $\operatorname{acl}(M \cup A)$. If $\varphi(x; \overline{b})$ has a realization $a \in \operatorname{acl}(\overline{b})$, then we are done. Otherwise, by weak minimality of T, any realization a of the formula is independent from $\overline{b}$. Let $p = \operatorname{stp}(\overline{b})$. Then

$$\mathfrak{C} \models \forall x \left[\varphi(x; \overline{b}) \leftrightarrow d_p \overline{y} \varphi(x; \overline{y}) \right],$$

and the formula $d_p \overline{y} \varphi(x; \overline{y})$ is definable over $\operatorname{acl}(\emptyset)$, so it is realized in M. □

The next lemma is true in *any* theory (not only weakly minimal ones).

Lemma 2.3. *Suppose that T has an infinite collection of models which are pairwise nonisomorphic and pairwise elementarily bi-embeddable, and $a \in \operatorname{acl}(\emptyset)$. Then the expansion $T_a := \operatorname{Th}(\mathfrak{C}, a)$ with a new constant naming a does not have the SB property. In fact, T_a also has an infinite collection of pairwise nonisomorphic, pairwise bi-embeddable models.*

Proof. Let $\{M_i : i < \omega\}$ be an infinite collection of models of T, pairwise nonisomorphic and pairwise bi-embeddable, and let n be the number of distinct realizations of $\operatorname{tp}(a)$. Then we claim that among any $n+1$ of the M_i's – say, $M_0, \ldots, M_n$ – there are two that are bi-embeddable as models of T_a. To see this, first pick elementary embeddings $f_k : M_0 \to M_k$ and $g_k : M_k \to M_0$ for every k with $1 \leq k \leq n$. Without loss of generality, every map $f_k \circ g_k$ fixes a, since we can replace f_k by $(f_k \circ g_k)^i \circ f_k$ for some i such that $(f_k \circ g_k)^{i+1}(a) = a$. By the pigeonhole principle, there must be two distinct $k, \ell \leq n$ such that g_k and g_ℓ map a onto the same element a'. Thus $f_k(a') = f_\ell(a') = a$, and so the maps

$$f_k \circ g_\ell : M_\ell \to M_k$$

and

$$f_\ell \circ g_k : M_k \to M_\ell$$

both fix a. Since $M_k \not\cong M_\ell$, they are not isomorphic as models of T_a either. □

3 Weakly minimal groups

In this section, we consider the relation between weakly minimal groups in a countable language and the SB property. We show that if T is any weakly minimal theory in which there is a definable weakly minimal abelian group with a certain kind of "generic" automorphism, then T does not have the SB property. One of the key lemmas is a variation of the Baire category theorem (Lemma 3.10).

Fact 3.1. *([8]) If $(G; \cdot, \ldots)$ is an $\emptyset$-definable weakly minimal group, then G has an $\emptyset$-definable abelian subgroup H of finite index.*

Throughout this section, we assume that $(G; +)$ is a weakly minimal *abelian* group which is $\emptyset$-definable in the *countable* theory T. The group G is equipped with all the definable structure induced from T, which may include additional structure that is not definable from the group operation alone. Fact 3.1 shows the assumption that G is abelian is not too strong.

We make the following additional assumptions:

1. G is saturated. We identify the set of all strong types over 0 with the set of points in $\overline{G} := G/G^\circ$, and we refer to subsets X of $\overline{G}$ as being dense, open, etc. if the corresponding subsets of the Stone space are.

2. The connected component G° is the intersection of the $\emptyset$-definable groups $G_0 > G_1 > G_2 > \ldots$, each of which is a subgroup of G of finite index.

3. $G_{i+1} \neq G_i$. (This assumption will be justified later.)

We let $\mathrm{Aut}(G)$ denote the group of all *elementary* bijections from G to G (not just the group of all group automorphisms), and we let $\mathrm{Aut}(\overline{G})$ be the group of all group automorphisms of $\overline{G}$ which are induced by maps in $\mathrm{Aut}(G)$.

We recall some important facts about the definable structure of a weakly minimal group G. The forking relation between generic elements is controlled by the action of the division ring D of definable quasi-endomorphisms of G (see [8]). Any nonzero quasi-endomorphism $d \in D$ is represented by a definable subgroup $S_d \leq G \times G$ which is an "almost-homomorphism," that is, the projection of S_d onto the first coordinate is a subgroup of finite index and the cokernel $\{g \in G : (0, g) \in S_d\}$ is finite. In our context, we may take S_d to be $\mathrm{acl}(0)$-definable or even 0-definable, so D is countable. From this point on, we fix some such 0-definable S_d representing each $d \in D$; the particular choice of S_d will not turn out to matter for our purposes.

If $d \in D \setminus \{0\}$ and H is any subgroup of $\mathrm{acl}(0)$ containing $\ker(S_d) \cup \mathrm{coker}(S_d)$, then S_d naturally induces an injective map $\overline{S_d}$ from a subgroup of $K \leq G/H$ of finite index into G/H; by extension, we think of this as d itself acting on K, and write this map as "$\overline{d}_H$" or simply "$\overline{d}$."

Note that $\overline{G}$ is a Polish group under the Stone topology. (The fact that the cosets of the groups G_i/G° form a base for the topology ensures that the space is separable.) Condition 3 implies that $\overline{G}$ is perfect (i.e. there are no isolated points). We will repeatedly use the fact that any $\varphi \in \mathrm{Aut}(\overline{G})$ (or indeed any image of an elementary embedding from G into itself) is continuous with respect to this topology.

Example 3.2. Let G be the direct product of ω copies of the cyclic group $\mathbb{Z}_p$, with its definable structure given by the group operation $+$ and unary predicates for each of the subgroups H_i consisting of all elements of g whose ith coordinate is zero. Then if we let G_i be the intersection of the groups $H_0, \ldots, H_{i-1}$, we are in the situation above, with $D \cong \mathbb{F}_p$. Note that although $|\mathrm{Aut}(\overline{G})| = 2^{\aleph_0}$, every $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$ has the property that $\varphi^{p-1} = \mathrm{id}$, so in the terminology of Definition 3.6 below, every automorphism is unipotent.

Now suppose that H is a *finite* 0-definable subgroup of G. Then if $\overline{G}_H$ denotes $G/(G^\circ + H)$, we can quotient by the projection map $\pi : \overline{G} \to \overline{G}_H$ to define a topology on $\overline{G}_H$. A crucial observation for what follows is that $\overline{G}_H$ is still a perfect Polish group. Note that the finiteness of H lets us conclude that $\overline{G}_H$ is perfect; if, say, $(H + G^\circ)/G^\circ$ were dense in $\overline{G}$, then the topology on $\overline{G}_H$ would be trivial.

Some more notation: fix some finite 0-definable $H \leq G$. If $\varphi \in \mathrm{Aut}(G)$, then $\varphi^*_H : \overline{G}_H \to \overline{G}_H$ is the corresponding automorphism of $\overline{G}_H$. We may write this as simply "φ^*" if H is understood. If g is an element of G or $\overline{G}$, then $\overline{g}_H$ is its image in $G/(G^\circ + H)$ under the natural quotient projection. If $d \in D$ and there is some 0-definable S_d representing d such that $\mathrm{coker}(S_d) \subseteq H$, then d^*_H (or d^*) is the corresponding partial function on $\overline{G}_H$. Note that in computing d^*_H, the particular choice of S_d representing d only affects the domain of d^*_H; two different choices of S_d result in partial maps on $\overline{G}_H$ which agree on their common domain, and this common domain is a subgroup of $\overline{G}_H$ of finite index. This motivates the following:

Definition 3.3. If f_1 and g_2 are two group homomorphisms from open subgroups $K_1, K_2 \leq \overline{G}_H$ into $\overline{G}_H$, then we write "$f_1 =^* f_2$" if there is some open subgroup K' of $K_1 \cap K_2$ such that $f_1 \upharpoonright K' = f_2 \upharpoonright K'$.

Definition 3.4.
1. If $D_0 \subseteq D$ is finite, then H is *good for* D_0 if H is a finite 0-definable subgroup of G containing $\mathrm{coker}(S_d)$ for every $d \in D_0$.
2. If $q \in D[x]$, then H is *good for* q if H is good for the set of coefficients of q.

Definition 3.5. If $X \subseteq \overline{G}$, then we say that $g + G^\circ$ is in the *D-closure of* X, or $\mathrm{cl}_D(X)$, if there are:
1. Elements $h_1 + G^\circ, \ldots, h_n + G^\circ$ of X,
2. Elements $d_1, \ldots, d_n$ of D, and
3. A subgroup $H \leq G$ which is good for $\{d_1, \ldots, d_n\}$,

such that $g_H = (d_1)^*_H(\overline{h_1}_H) + \ldots + (d_n)^*_H(\overline{h_n}_H)$.
The set $X \subseteq \overline{G}$ is *D-closed* if $X = \mathrm{cl}_D(X)$.

If $q = \Sigma_{i \leq n} d_i x^i$ is a polynomial in x over D, and $H \leq G$ is good for q, then

$$q^*_H \varphi := \Sigma_{i \leq n} (d_i)^*_H \circ (\varphi^*_H)^i.$$

Note that q^*_H is a continuous group map from a finite-index subgroup of $\overline{G}_H$ into $\overline{G}_H$, and that $(q + r)^*_H =^* q^*_H + r^*_H$ and $(q \cdot r)^*_H =^* q^*_H \circ r^*_H$.

Definition 3.6.

1. $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$ is *unipotent* if there is some nonzero $n \in \omega$ such that $\overline{\varphi}^n = \mathrm{id}$.

2. $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$ is *weakly generic* if for every $q(x) \in D[x] \setminus \{0\}$ and every $H \leq G$ which is good for q, the map $q^*_H\varphi$ is not identically zero on its domain.

3. $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$ is *everywhere generic* if for every $q(x) \in D[x] \setminus \{0\}$, for every $H \leq G$ which is good for q, and for every nonempty open $U \subseteq \mathrm{dom}(q^*_H\varphi)$, the map $q^*_H\varphi \restriction U$ is not identically zero.

Proposition 3.7. *If $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$ is weakly generic, then $\overline{\varphi}$ is everywhere generic.*

Proof. Suppose that $\overline{\varphi}$ is *not* everywhere generic, as witnessed by $q \in D[x] \setminus \{0\}$, H, and a nonempty open $U \subseteq \mathrm{dom}(q^*_H\varphi)$ such that $q^*_H\varphi \restriction U$ is identically zero. Say $q = x^m \cdot q_0$, where q_0 has a nonzero constant term; then since $(\varphi^*_H)^m$ is an injective group homomorphism, $(q_0)^*_H\varphi \restriction (\varphi^*_H)^m(U)$ is identically zero and $(\varphi^*_H)^m(U)$ is open; so we may assume that x does not divide q. Write $q = d_k x^k + \ldots + d_0$, where $d_i \in D$. Since $d_0 \neq 0$, it follows that if $r = d_0^{-1} d_k x^k - \ldots - d_0^{-1} d_1 x$, then there is some nonempty open $V \subseteq \overline{G}_H$ such that $(d_0^{-1})^*(V) \subseteq U$ and $r^*_H\varphi \restriction V = \mathrm{id}_V$.

Without loss of generality, $V = (g + G_n + H)/(G^\circ + H)$ for some $g \in G$ and some $n < \omega$, and (shrinking V if necessary) we may also assume that $(G_n + H)/(G^\circ + H) \subseteq \mathrm{dom}(r^*_H\varphi)$. For any $h \in (G_n + H)/(G^\circ + H)$, there are $g_1, g_2 \in V$ such that $h = g_1 + g_2$, so it follows that $r^*_H\varphi \restriction (G_n + H)/(G^\circ + H)$ is the identity map. For any $\ell \in \omega$, the map $(r^\ell)^*_H\varphi$ induces a group homomorphism from a subgroup of $G/(G_n + H)$ into $G/(G_n + H)$, and since $G/(G_n + H)$ is finite, there are numbers $\ell < m < \omega$ such that the image of $(r^\ell - r^m)^*_H\varphi$ is contained in $(G_n + H)/(G^\circ + H)$. Let $s = (r^\ell - r^m)^2$. Note that since x divides r, the polynomial s is nonzero, but $s^*_H\varphi$ is identically zero on its domain; so φ is not weakly generic.

□

Question 3.8. *If $\mathrm{Aut}(\overline{G})$ contains a non-unipotent element, does $\mathrm{Aut}(\overline{G})$ necessarily contain a weakly generic element?*

Definition 3.9. Let S be a Polish space. A continuous function $f : S^k \to S$ is *nondegenerate* if there is some i ($1 \leq i \leq k$) such that for any elements $a_i, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in S$, the function

$$f_{\overline{a}}(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$$

is a homeomorphism.

The next lemma is a version of the Baire category theorem. (To get the usual Baire category theorem, let $f_0 : S \to S$ be the identity map.)

Lemma 3.10. *Suppose that S is a perfect Polish space and $\langle f_i : i \in \omega\rangle$ is a countable collection of continuous, nondegenerate functions, with $f_i : S^{k(i)} \to S$, and $\langle H_\ell : \ell \in \omega\rangle$ is a collection of nowhere dense subsets of S. Then there is a nonempty perfect set of elements $\langle a_\sigma : \sigma \in 2^{\aleph_0}\rangle$ of S such that for every $i, \ell \in \omega$ and every set $\{\sigma_1, \ldots, \sigma_{k(i)}\}$ of* $k(i)$ distinct *elements of* $2^{\aleph_0}$, *$f_i(a_{\sigma_1}, \ldots, a_{\sigma_{k(i)}}) \notin H_\ell$.*

Proof. First, we set some notation. Let $\mathcal{O}$ be the set of all *nonempty* open subsets of S. A *condition* is a function $F : D \to \mathcal{O}$, where D is some finite, downward-closed subset of $2^{<\omega}$, such that

1. If $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in D$, then $F(s^\frown\langle 0\rangle) \cap F(s^\frown\langle 1\rangle) = \emptyset$; and
2. If s is an initial segment of t, then $F(t) \subseteq F(s)$.

Given two conditions F, F', we write $F \le F'$ if $\mathrm{dom}(F) \subseteq \mathrm{dom}(F')$ and for any $s \in \mathrm{dom}(F)$, $F'(s) \subseteq F(s)$.

A *viable triple* is an ordered triple $\langle i, \ell, (s_1, \ldots, s_k)\rangle$ such that $i, \ell \in \omega$, the s_j's are pairwise incompatible elements of $2^{<\omega}$, and $k = k(i)$. Let

$$\left\{\langle i(t), \ell(t), (s_1^t, \ldots, s_{k(t)}^t)\rangle : t \in \omega\right\}$$

be an enumeration of all viable triples.

We construct an increasing sequence of conditions $\langle F(t) : t < \omega\rangle$ by induction on t. As a base case, let $F(0)$ be the function with domain $\{\langle\rangle\}$ such that $F(\langle\rangle) = S$. For the induction step, suppose that we have picked $F(t)$.

Claim 3.11. We can pick $F(t+1) \ge F(t)$ such that

$$\{s_1^t, \ldots, s_{k(t)}^t\} \subseteq \mathrm{dom}(F(t+1))$$

and

$$f_{i(t)}(F(t+1)(s_1^t), \ldots, F(t+1)(s_{k(t)}^t)) \cap H_{\ell(t)} = \emptyset.$$

Proof. First, since S is perfect, we can pick a condition $F' \ge F(t)$ such that $\{s_1^t, \ldots, s_{k(t)}^t\} \subseteq \mathrm{dom}(F')$. By the fact that $f_{i(t)}$ is nondegenerate and $H_{\ell(t)}$ is nowhere dense, there is a tuple $(a_1, \ldots, a_{k(t)})$ such that $a_m \in F'(s_m^t)$ and $f_{i(t)}(a_1, \ldots, a_{k(t)}) \notin \overline{H_{\ell(t)}}$. Pick an open neighborhood U of $f_{i(t)}(a_1, \ldots, a_{k(t)})$ such that $U \cap \overline{H_{\ell(t)}} = \emptyset$. By continuity, $f_{i(t)}^{-1}(U)$ is open, and it contains $(a_1, \ldots, a_{k(t)})$. Therefore, there is a condition $F(t+1) \ge F'$ such that $a_m \in F(t+1)(s_m^t)$ and

$$f_{i(t)}(F(t+1)(s_1^t), \ldots, F(t+1)(s_{k(t)}^t)) \subseteq U,$$

so this $F(t+1)$ works.

□

Pick $F(t+1)$ by induction as in the Claim. Note that it follows that

$$\bigcup_{t \in \omega} \mathrm{dom}(F(t)) = 2^{<\omega},$$

since every $s \in 2^{<\omega}$ is in a viable triple. For any $\sigma \in 2^\omega$, let

$$\widehat{F}(\sigma) = \bigcap\{F(t)(\sigma \restriction m) : t \in \omega \text{ and } \sigma \restriction m \in \mathrm{dom}(F(t))\}.$$

It follows from the properties of the conditions $F(t)$ that $\widehat{F}(\sigma)$ is always well-defined and nonempty and that if $\sigma \neq \tau$, then $\widehat{F}(\sigma) \cap \widehat{F}(\tau) = \emptyset$. Finally, pick elements $a_\sigma \in \widehat{F}(\sigma)$ for every $\sigma \in 2^\omega$. The fact that these a_σ's work follows from the way we enumerated the viable triples and the Claim above.

□

We need one more simple lemma before proving the main theorem of this section.

Lemma 3.12. *Suppose that T is* any *stable theory, $M \models T$, $\theta(x)$ is a weakly minimal formula over $\emptyset$ in T, and $A \subseteq \theta(\mathfrak{C})$. Then if $M' = \mathrm{acl}(M \cup A)$ and $M' \models T$,*

$$\theta(M') \subseteq \mathrm{acl}(\theta(M) \cup A).$$

Proof. Suppose $b \in \theta(M')$ and $b \in \mathrm{acl}(M \cup \{a_1, \ldots, a_n\})$, where $a_i \in A$ and n is minimal. Minimality of n implies that $\{a_1, \ldots, a_n\}$ is independent over M and b is interalgebraic with a_n over $M \cup \{a_1, \ldots, a_{n-1}\}$. Note that $\mathrm{tp}(ba_1 \ldots a_n/M)$ is finitely satisfiable in $\theta(M)$, and therefore

$$ba_1 \ldots a_n \mathop{\smile\!\!\!\!\!\!\mid}_{\theta(M)} M.$$

So

$$U(ba_1 \ldots a_n/\theta(M)) = U(ba_1 \ldots a_n/M) = n,$$

and therefore b is interalgebraic with a_n over $\theta(M) \cup \{a_1, \ldots, a_{n-1}\}$. □

Theorem 3.13. *Suppose that T is weakly minimal and countable, and that G is a weakly minimal abelian group which is $\emptyset$-definable in T. If $\mathrm{Aut}(\overline{G})$ contains a weakly generic map $\overline{\varphi}$, then T has an infinite collection of pairwise nonisomorphic, pairwise elementarily bi-embeddable models.*

Proof. First we note that if $\mathrm{Aut}(\overline{G})$ contains a weakly generic map, then $\overline{G}$ must be infinite, so we can pick the definable subgroups $G_0 > G_1 > \ldots$ so that $G_{i+1} \neq G_i$. This justifies assumption 3 at the beginning of the section and implies that $\overline{G}$ is a perfect topological space.

The key to our proof is that we can use the fact that $|\mathrm{Aut}(\overline{G})| = 2^{\aleph_0}$ to construct bi-embeddable models M and N of T via chains of length $2^{\aleph_0}$ where at each successor stage we kill one potential isomorphism between $\overline{G}(M)$ and $\overline{G}(N)$. We call this a Dushnik-Miller type argument since it recalls the idea of the proof of Theorem 5.32 in [2].

As usual, let D be the division ring of definable quasi-endomorphisms of G. Fix a weakly generic $\overline{\varphi} \in \mathrm{Aut}(\overline{G})$, which is also everywhere generic by Proposition 3.7. Also pick some $f \in \mathrm{Aut}(\mathfrak{C})$ such that $\overline{f \restriction G(\mathfrak{C})} = \overline{\varphi}$.

Since $\overline{G}$ is separable, $|\mathrm{Aut}(\overline{G})| \leq 2^{\aleph_0}$. Therefore we can pick a sequence $\{(h_\alpha, i(\alpha), j(\alpha)) : \alpha < 2^{\aleph_0}\}$ listing triples in $\mathrm{Aut}(\overline{G}) \times \omega \times \omega$ in such a way that for any $\alpha < 2^{\aleph_0}$,

(A) $i(\alpha) < j(\alpha)$, and

(B) For any $i_0 < j_0 < \omega$ and any $h \in \mathrm{Aut}(\overline{G})$, there is some β such that $\alpha < \beta < 2^{\aleph_0}$ and $(h_\beta, i(\beta), j(\beta)) = (h, i_0, j_0)$.

Next, we will define models M^ℓ_α of T and subsets X^ℓ_α of $\overline{G}$ for every $\ell < \omega$ and $\alpha < 2^{\aleph_0}$ by recursion on $\alpha < 2^{\aleph_0}$ such that:

1. $|M^\ell_\alpha \cup X^\ell_\alpha| \leq |\alpha| + \aleph_0$;
2. $M^0_\alpha \succ M^1_\alpha \succ M^2_\alpha \succ \ldots$ and $f(M^\ell_\alpha) \prec M^{\ell+1}_\alpha$;
3. If $\alpha < \beta$, then $M^\ell_\alpha \prec M^\ell_\beta$ and $X^\ell_\alpha \subseteq X^\ell_\beta$;
4. $\overline{G}(M^\ell_\alpha) \cap X^\ell_\alpha = \emptyset$;
5. If $\alpha < \beta$, then at least one of the following holds:
   (C) There is some $a \in \overline{G}(M^{i(\alpha)}_\beta)$ such that $h_\alpha(a) \in X^{j(\alpha)}_\beta$; or
   (D) There is some $b \in \overline{G}(M^{j(\alpha)}_\beta)$ such that $h^{-1}_\alpha(b) \in X^{i(\alpha)}_\beta$.

Once we have the models M^ℓ_α, we can let $M^\ell = \bigcup_{\alpha < 2^{\aleph_0}} M^\ell_\alpha$. The M^ℓ's are pairwise bi-embeddable (via inclusions and iterates of f), and properties 4 and 5 ensure that if $\ell \neq k$ then there is no $h \in \mathrm{Aut}(\mathfrak{C})$ mapping $\overline{G}(M^\ell)$ onto $\overline{G}(M^k)$, so *a fortiori* there is no such h mapping M^ℓ onto M^k. Thus $\{M^\ell : \ell < \omega\}$ will be the models we are looking for.

For the base case $\alpha = 0$, we can pick some countable $M \models T$ such that $f(M) \subseteq M$ and let $M^\ell_0 = M$ and $X^\ell_0 = \emptyset$. At limit stages we take unions, and there are no problems. So we only have to deal with the successor stage, and suppose we have M^ℓ_α and X^ℓ_α as above.

To set some more notation, if $Z \subset \overline{G}$ and $H \leq G$, then we let

$$Z_H = \left\{ h \in \overline{G}_H : \exists g \in G \; [g + G^\circ \in Z \text{ and } h = g + (G^\circ + H)] \right\}.$$

Here are two possible situations which we will consider:

($*$) There is a nonempty clopen *subgroup* $K \leq \overline{G}$ with the following property: for every $q(x) \in D[x]$ such that $x | q$, every nonempty open $U \subseteq K$, and every finite 0-definable $H \leq G$ which is good for q, there is a $g \in U_H$ such that either $g \notin \mathrm{dom}(q^*_H \varphi)$ or $(h_\alpha)^*_H(g) \neq q^*_H \varphi(g)$.

($\dagger$) For every $q(x) \in D[x]$, every nonempty open $U \subseteq \overline{G}$, and every finite 0-definable $H \leq G$ which is good for q, there is a $g \in U_H$ such that either $g \notin \mathrm{dom}(q^*_H \varphi)$ or $(h^{-1}_\alpha)^*_H(g) \neq q^*_H \varphi(g)$.

Claim 3.14. Either ($*$) holds or ($\dagger$) holds (or possibly both).

Proof. The failure of ($\dagger$) gives us a nonempty open $U \subseteq \overline{G}$, a $q(x) \in D[x]$, and $H \leq G$ which is good for q such that for every $g \in U_H$, $(h^{-1}_\alpha)^*_H(g) = q^*_H \varphi(g)$. Let K be the subgroup of $\overline{G}$ generated by U. The continuity of the group

operation implies that K is open, in fact clopen, and since $(h_\alpha^{-1})^*_H$ and $q^*_H\varphi$ are both homomorphisms from $\overline{G}_H$ into itself,

$$(h_\alpha^{-1})^*_H \upharpoonright K_H = q^*_H\varphi \upharpoonright K_H. \tag{1}$$

Let $K' = h_\alpha^{-1}(K)$, so K' is another clopen subgroup of $\overline{G}$. By the failure of $(*)$, there is an $r(x) \in D[x]$ such that $x|r$, some nonempty open $V \subseteq K'$, and some $H' \leq G$ good for r such that for every $g \in V_H$,

$$(h_\alpha)^*_{H'}(g) = r^*_{H'}\varphi(g). \tag{2}$$

Note that equations 1 and 2 remain true if we replace either H or H' with any larger finite 0-definable subgroup of G, such as $\widehat{H} := H + H'$. For any $g \in (h_\alpha)^*_{\widehat{H}}(V_{\widehat{H}})$, note that

$$g \in (h_\alpha)^*_{\widehat{H}}(K'_{\widehat{H}}) = K_{\widehat{H}},$$

and so by equations 1 and 2,

$$g = (h_\alpha)^*_{\widehat{H}}\left((h_\alpha^{-1})^*_{\widehat{H}}(g)\right) = (r \cdot q)^*_{\widehat{H}}\varphi(g).$$

Note that since $x|(r \cdot q)$, the polynomial $r \cdot q - 1$ is nonzero, so we have a contradiction to the fact that φ is everywhere generic. $\square$

Now we return to the main proof and argue by cases.

Case 1: $(*)$ holds.

Fix a nonempty clopen $K \leq \overline{G}$ witnessing $(*)$.

Claim 3.15. There is an element $a \in K$ such that

(E) For any $b \in \overline{G}(M_\alpha^{j(\alpha)})$, any $q \in D[x]$ such that $x|q$, and any $H \leq G$ which is good for q, if $a_H \in \mathrm{dom}(q^*_H\varphi)$, then

$$(h_\alpha)^*_H(a_H) - q^*_H\varphi(a_H) \neq b_H;$$

and

(F) For any $c \in \mathrm{cl}_D(\bigcup_{\ell<\omega}(\overline{G}(M_\alpha^\ell) \cup X_\alpha^\ell))$, any nonzero $q \in D[x]$, and any $H \leq G$ good for q, if $a_H \in \mathrm{dom}(q^*_H\varphi)$, then

$$q^*_H\varphi(a_H) \neq c_H.$$

Proof. For any $q \in D[x]$ and any $H \leq G$ which is good for q, we define two subgroups $K_{q,H}$ and $K'_{q,H}$ of K as follows. If $x|q$, then let

$$K_{q,H} = \pi_H^{-1}\left[\ker((h_\alpha)^*_H - q^*_H\varphi)\right] \cap K,$$

where $\pi_H : \overline{G} \to \overline{G}_H$ is the natural projection map. Note that $(h_\alpha)^*_H - q^*_H$ is a continuous group map, so its kernel is a closed subgroup of $\overline{G}_H$, and therefore $K_{q,H}$ is a closed subgroup of $\overline{G}$. If x does not divide q, we let $K_{q,H} = \{0\}$. By assumption $(*)$, $K_{q,H}$ is always nowhere dense. If $q \neq 0$, then let

$$K'_{q,H} = \pi_H^{-1}\left[\ker(q^*_H\varphi)\right] \cap K,$$

and if $q = 0$, then let $K'_{q,H} = \{0\}$. As before, $K'_{q,H}$ is a closed subgroup of $\overline{G}$, and the genericity of $\overline{\varphi}$ implies that $K'_{q,H}$ is nowhere dense.

So by Lemma 3.10 applied to the nondegenerate map $(x, y) \mapsto x - y$, there is a collection $\langle a_\sigma : \sigma < 2^{\aleph_0} \rangle$ of elements of K such that for any two distinct $\sigma, \tau < 2^{\aleph_0}$, $a_\sigma - a_\tau$ does not lie in any of the groups $K_{q,H}$ or $K'_{q,H}$. Now since

$$|\mathrm{cl}_D \left(\bigcup_{\ell < \omega} (\overline{G}(M^\ell_\alpha) \cup X^\ell_\alpha) \right) | < 2^{\aleph_0},$$

for any $q \in D[x]$ there are fewer than $2^{\aleph_0}$ choices of $\sigma < 2^{\aleph_0}$ such that a_σ belongs to the $K_{q,h}$-coset or the $K'_{q,H}$-coset of some element of $\mathrm{cl}_D(\bigcup_{\ell<\omega}(\overline{G}(M^\ell_\alpha) \cup X^\ell_\alpha))$. Since the cofinality of $2^{\aleph_0}$ is uncountable, there is some $\sigma < 2^{\aleph_0}$ such that for *any* $q \in D[x]$ and any $h \in \mathrm{cl}_D(\bigcup_{\ell<\omega}(\overline{G}(M^\ell_\alpha) \cup X^\ell_\alpha))$, $a_\sigma - h \notin (K_{q,H} \cup K'_{q,H})$. Let $a = a_\sigma$; this works.

$\square$

Pick a as in the Claim above and pick $g \in G(\mathfrak{C})$ such that $g + G^\circ = a$. If $\ell \leq i(\alpha)$, then we let

$$M^\ell_{\alpha+1} = \mathrm{acl}(M^\ell_\alpha \cup \{f^n(g) : n < \omega\}),$$

and if $\ell > i(\alpha)$, we let

$$M^\ell_{\alpha+1} = \mathrm{acl}(M^\ell_\alpha \cup \{f^n(g) : (\ell - i(\alpha)) \leq n < \omega\}).$$

If $\ell \neq j(\alpha)$, the we let $X^\ell_{\alpha+1} = X^\ell_\alpha$, and let $X^{j(\alpha)}_{\alpha+1} = X^{j(\alpha)}_\alpha \cup \{h_\alpha(a)\}$.

We check this works. Note that condition 5 (C) holds by definition, and conditions 1 through 3 are automatic. For Condition 4, we first check that $\overline{G}(M^\ell_{\alpha+1}) \cap X^\ell_\alpha = \emptyset$. First note that by Lemma 3.12,

$$\overline{G}(M^\ell_{\alpha+1}) = \mathrm{cl}_D \left(\overline{G}(M^\ell_\alpha) \cup \{\varphi^n(a) : n < \omega\} \right).$$

So if Condition 4 fails, then there is some $b \in \overline{G}(M^\ell_\alpha)$, some $q \in D[x]$, and some $H \leq G$ good for q such that

$$b_H + q^*_H \varphi(g_H) \in (X^\ell_\alpha)_H,$$

but since $a_H = g_H$, this contradicts condition (F) above. The only other way that 4 could fail is if $h_\alpha(a) \in \overline{G}(M^{j(\alpha)}_{\alpha+1})$, or equivalently (by Lemma 3.12), there is some $q \in D[x]$ such that $x|q$, some $H \leq G$ good for q, and some $b \in \overline{G}(M^{j(\alpha)}_\alpha)$ such that

$$(h_\alpha(a))^*_H = (h_\alpha)^*_H(a_H) = b_H + q^*_H \varphi(a_H),$$

but this contradicts (E).

Case 2: (†) holds.

Exactly like in Case 1, we have:

Claim 3.16. There is an element $b \in \overline{G}$ such that

(E') For any $a \in \overline{G}(M_\alpha^{i(\alpha)})$, any $q \in D[x]$, and any $H \leq G$ which is good for q, if $b_H \in \mathrm{dom}(q_H^* \varphi)$, then

$$(h_\alpha^{-1})_H^*(b) - q_H^* \varphi(b_H) \neq a_H;$$

and

(F') For any $c \in \mathrm{cl}_D(\bigcup_{\ell<\omega}(\overline{G}(M_\alpha^\ell) \cup X_\alpha^\ell))$, any nonzero $q \in D[x]$, and any $H \leq G$ good for q, if $b_H \in \mathrm{dom}(q_H^* \varphi)$, then

$$q(\varphi)(b_H) \neq c_H.$$

Then pick $g \in G(\mathfrak{C})$ such that $g + G^\circ = b$, and let

$$M_{\alpha+1}^\ell = \mathrm{acl}(M_\alpha^\ell \cup \{f^n(g) : n < \omega\}),$$

$$N_{\alpha+1} = \mathrm{acl}(N_\alpha \cup \{f^n(h) : n < \omega\}),$$

let $X_{\alpha+1}^\ell = X_\alpha^\ell$ if $\ell \neq i(\alpha)$, and let $X_{\alpha+1}^{i(\alpha)} = X_\alpha^{i(\alpha)} \cup \{h_\alpha^{-1}(b)\}$.

Just as before, it can be checked that these sets satisfy conditions 1 through 5.

□

Remark 3.17. Since for the base case of our construction, we let the models M_0^ℓ (for $\ell < \omega$) all be equal, it is worth pointing out why the models M^ℓ we eventually construct are not all equal. To see this, note that the identity map from $\overline{G}$ to $\overline{G}$ will be listed (infinitely often) as some h_α, and so the fact that f has independent orbits implies that $(*)$ holds (with $K = \overline{G}$). Thus at stage α, we ensure that $M^{i(\alpha)} \neq M^{j(\alpha)}$.

4 Weakly minimal theories

In this section, we return to the general context of weakly minimal theories and prove Theorem 1.4. We assume throughout this section that T is a weakly minimal theory, and "$a \in M$" means that a is in the home sort of M. These assumptions imply that for any $a \in M$, the type $\mathrm{tp}(a/\emptyset)$ is *minimal* (that is, has U-rank 1), though it is not necessarily weakly minimal.

Definition 4.1. Let $p \in S(A)$ be a minimal type.

1. The type p *has bounded orbits over* B if there is some $n < \omega$ such that for every $f \in \mathrm{Aut}(\mathfrak{C}/B)$, there are $i < j \leq n$ such that $f^i(p) = f^j(p)$. Otherwise, p *has unbounded orbits* over B.

2. The type p *has dependent orbits over* B if for every $f \in \mathrm{Aut}(\mathfrak{C}/B)$, there is an $n < \omega$ such that

$$f^n(p) \not\perp^a p \otimes f(p) \otimes \ldots \otimes f^{n-1}(p).$$

Otherwise, p *has an independent orbit over* B.

3. If $p \in S(\mathrm{acl}(\emptyset))$, then we say p has bounded orbits (or dependent orbits) if it has bounded (dependent) orbits over $\emptyset$.

Remark 4.2. The minimal type $p \in S(\mathrm{acl}(\emptyset))$ has an independent orbit if and only if there is an $f \in \mathrm{Aut}(\mathfrak{C})$ such that for *any* choice of realizations $\langle a_i : i < \omega \rangle$ of the types $f^i(p)$, the set $\{a_i : i < \omega\}$ is independent.

Question 4.3. *1. If p has unbounded orbits, then does p necessarily have an independent orbit?*

2. If $p \in S(\mathrm{acl}(\emptyset))$ has dependent orbits and $g \in \mathrm{Aut}(\mathfrak{C})$, then does $g(p)$ also have dependent orbits?

Note that in the terminology above, if p is the generic type of a weakly minimal, locally modular group G defined over $\mathrm{acl}(\emptyset)$, then $\mathrm{Aut}(G)$ does not have a weakly generic element if and only if for *every* generic type q of G has dependent orbits.

If $p \in S(\mathrm{acl}(\emptyset))$ is minimal and $M \models T$, let "$\dim(p, M)$" mean the dimension of $p(M)$ as a pregeometry.

Definition 4.4. An elementary map $f : M \to N$ between models of T is called *dimension-preserving* if for any minimal $p \in S(\mathrm{acl}(\emptyset))$, $\dim(p, M) = \dim(f(p), N)$.

Theorem 4.5. *Any two models M, N of T are isomorphic if and only if there is a dimension-preserving map $f : M \to N$.*

Proof. Left to right is obvious, since any isomorphism is dimension-preserving. For the converse, suppose that $f : M \to N$ is dimension-preserving.

If $A \subseteq M$, we say that A is *type-closed (in M)* if $\mathrm{acl}(\emptyset) \subseteq A$ and whenever $a \in A$, $a' \in M$, and $\mathrm{stp}(a') = \mathrm{stp}(a)$ is minimal, then $a' \in A$.

If $A \subseteq M$ and $B \subseteq N$, we call an elementary map $g : A \to B$ *closed* if its domain is type-closed in M and its image is type-closed in N.

If $A \subseteq \mathfrak{C}$ and $p \in S(\mathrm{acl}(\emptyset))$, then define $k(p, A) \in \omega$ to be the largest k (if one exists) such that $p^{(k)}$ is almost orthogonal to $\mathrm{tp}(A/\mathrm{acl}(\emptyset))$. Note that A is independent from any Morley sequence in p of length at most $k(p, A)$ (if it exists), and A is independent from every Morley sequence in p if $k(p, A)$ does not exist.

Claim 4.6. Suppose that $A \subseteq M$ is type-closed, $p \in S(\mathrm{acl}(\emptyset))$ is minimal, and $k(p, A)$ exists. Then for any Morley sequence $I_0 \subseteq M$ in p of length $k(p, A)$, $p(M) \subseteq \mathrm{acl}(A \cup I_0)$.

Proof. By definition of $k(p, A)$, there is some $a \models p|I_0$ such that $a \not\perp_{I_0} A$. Pick such an a and a finite set $A_0 \cup \{c\} \subseteq A$ such that $I_0 a \not\perp A_0 c$. We may assume that the size of $A_0 \cup \{c\}$ is minimal, so that A_0 is independent from any Morley sequence in p of length $k(p, A)+1$. By the weak minimality of T and stationarity, if $q = \mathrm{stp}(c)$, then *any* realization of $p|I_0$ is interalgebraic over $I_0 \cup A_0$ with some realization of $q|A_0$. So any $b \in p(M) \setminus \mathrm{acl}(I_0)$ is interalgebraic over $A_0 \cup I_0$ with some $d \in q(M)$, and the fact that A is type-closed implies that $d \in A$. $\square$

It follows from Claim 4.6 (when $k(p, A) = 0$) that if $A \subseteq M$ is type-closed, then so is $\mathrm{acl}(A)$. Thus, any elementary $h : \mathrm{acl}(A) \to \mathrm{acl}(B)$ extending a closed map $g : A \to B$ is closed.

Visibly, the restriction $\sigma := f \upharpoonright \mathrm{acl}(\emptyset)$ is closed, and the union of an increasing sequence of closed maps is closed. Thus, to conclude that M and N are isomorphic, it suffices by Zorn's Lemma and symmetry to show that if $g : A \to B$ extending σ is closed, and $p \in S(\mathrm{acl}(\emptyset))$ is minimal, then there is a closed h extending g, whose domain contains $A \cup p(M)$.

Fix such a g and p. Note that $k(p, A)$ exists if and only if $k(g(p), B)$ exists and when they exist, they are equal. There are three cases:

Case 1: $k(p, A)$ does not exist. Let $I \subseteq M$ be any maximal Morley sequence in p, and $J \subseteq N$ be any maximal Morley sequence in $g(p)$. Since g extends σ, g is dimension preserving, so $|I| = |J|$. Let $j : I \to J$ be any bijection. Then $g \cup j$ is elementary, and any elementary map $h : \mathrm{acl}(A \cup I) \to \mathrm{acl}(B \cup J)$ extending $g \cup j$ will be closed.

Case 2: $k(p, A)$ exists, but $\dim(p, M) \leq k(p, A)$. Then again, $\dim(p, M) = \dim(g(p), N)$, and for any bijection j between bases for $p(M)$ and $g(p)(M)$, there is a closed $h : \mathrm{acl}(AI) \to \mathrm{acl}(BJ)$ extending $g \cup j$.

Case 3: $k(p, A)$ exists, and $\dim(p, M) > k(p, A)$. Let $I_0 \subseteq p(M)$ and $J_0 \subseteq g(p)(N)$ be Morley sequences of length $k(p, A)$ in p and $g(p)$, respectively, and let $j : I_0 \to J_0$ be any bijection. Again, $g \cup j$ is elementary and any elementary $h : \mathrm{acl}(A \cup I_0) \to \mathrm{acl}(B \cup J_0)$ extending $g \cup j$ is closed. But $p(M) \subseteq \mathrm{acl}(A \cup I_0)$ and $g(p)(M) \subseteq \mathrm{acl}(B \cup J_0)$ by Claim 4.6, so any such h suffices.

□

Corollary 4.7. *If T is weakly minimal and every minimal type $p \in S(\mathrm{acl}(\emptyset))$ has dependent orbits, then T has the SB property.*

Proof. If M and N are models of such a theory and $f : M \to N$ and $g : N \to M$ are elementary embeddings, then f must be dimension-preserving. So by Theorem 4.5, $M \cong N$. □

The next result is from the first author's thesis ([3]), where types as in the hypothesis were called "nomadic."

Theorem 4.8. *If T is stable and there is a stationary regular type $p \in S(A)$ and $f \in \mathrm{Aut}(\mathfrak{C})$ such that the types $\{f^i(p) : i < \omega\}$ are pairwise orthogonal, then T has an infinite collection of models that are pairwise nonisomorphic and pairwise not bi-embeddable.*

Proof. Just for simplicity of notation, we will assume that T is countable and superstable, but the same argument works in general if we lengthen our Morley sequences a bit and replace the $\mathbf{F}^a_{\aleph_0}$-prime models by $\mathbf{F}^a_\kappa$-prime models for some suitably large κ. We will use without proof some well-known facts about a-prime models which are proved in section 1.4 of [8] and in chapter IV of [9] (in the latter reference, they are called "$\mathbf{F}^a_{\aleph_0}$-prime models").

Pick p and f as in the hypothesis. Since T is superstable, we may assume that A is countable. Let $A_i = f^i(p)$ and $p_i = f^i(p)$ (which is in $S(A_i)$), and let

$B = \bigcup_{i<\omega} A_i$. Pick sequences $\langle I_i^j : i, j < \omega \rangle$ such that I_i^j is a Morley sequence in $p_i|B$ of length $\aleph_{i+j+1}$ and

$$I_i^j \underset{B}{\perp} \bigcup_{k \neq i} I_k^j.$$

For each $j < \omega$, let M_j be an a-prime model over $B \cup \bigcup_{i<\omega} I_i^j$.

By using a-primeness and iterating the map f, it is straightforward to check that the models $\langle M_j : j < \omega \rangle$ are pairwise bi-embeddable. To prove they are nonisomorphic, we set some notation. If $p \in S(C)$ is a regular type and $C \subseteq M \models T$, then $\dim(p, M)$ is the cardinality of a maximal Morley sequence in p inside M; as noted in section 1.4.5 of [8], this is well-defined for any model M.

Claim 4.9. If $q \in S(C)$ is any regular type and M is an a-prime model over C, then $\dim(q, M) \leq \aleph_0$.

Proof. This is a special case of Theorem IV.4.9(5) from [9]. □

Claim 4.10. If $q \in S(C)$ is any regular stationary type over a countable set $C \subseteq M_j$, then either $\dim(q, M_j) \leq \aleph_0$ or $\dim(q, M_j) \geq \aleph_{j+1}$.

Proof. Case 1: For some $i < \omega$, $q \not\perp p_i$.

Pick some N which is a-prime over $B \cup C$, and by primeness we may assume $N \prec M_j$. Since N is an a-model, $q|N$ is domination equivalent to $p_i|N$, and so $\dim(q|N, M_j) = \dim(p_i|N, M_j)$. If $J \subseteq N$ is a maximal Morley sequence in p_i, then each $c \in J$ forks with C over B, so $|J| \leq \aleph_0$; therefore $\dim(p_i, N) \leq \aleph_0$. By Lemma 1.4.5.10 of [8],

$$\dim(p_i, M_j) = \dim(p_i, N) + \dim(p_i|N, M_j);$$

so since $\dim(p_i, M_j) \geq \aleph_{i+j+1}$, we must have $\dim(p_i|N, M_j) \geq \aleph_{i+j+1}$. So $\dim(q, M_j) \geq \dim(q|N, M_j) = \dim(p_i|N, M_j) \geq \aleph_{j+1}$.

Case 2: For every $i < \omega$, $q \perp p_i$.

Pick some finite $D \subseteq C$ such that q does not fork over D. If $J \subseteq M_j$ is a maximal Morley sequence in q, then by standard forking calculus we can find a subset $\widetilde{J}$ of J such that $\widetilde{J} \perp_D B$, $\widetilde{J} \perp_{BD} I^j_{<\omega}$, and $|J \setminus \widetilde{J}| \leq \aleph_0$. Since D is finite, the model M_j is a-prime over $B \cup D \cup I^j_{<\omega}$, and so by Claim 4.9, $|\widetilde{J}| \leq \aleph_0$. Therefore $|J| \leq \aleph_0$ and we are done.

□

If $j < k < \omega$, then the nonisomorphism of M_j and M_k follows from the previous claim plus:

Claim 4.11. $\dim(p, M_j) = \aleph_{j+1}$.

Proof. Suppose $J \subseteq M_j$ is a maximal Morley sequence in p. Without loss of generality, $J \supseteq I_0^j$, so $|J| \geq \aleph_{j+1}$. Since B is countable, there is a countable $J_0 \subseteq J$ such that $J_1 := J \setminus J_0$ is independent over B. Since p is orthogonal to every type p_i for $i > 0$, J_1 is independent over $B \cup \bigcup_{0<i<\omega} I_i^j$. So if $J_2 =$

$J_1 \setminus I_0^j$, then J_2 is Morley over $B \cup \bigcup_{i<\omega} I_i^j$. But since M_j is a-constructible over $B \cup \bigcup_{i<\omega} I_i^j$, it follows (by Claim 4.9) that $|J_2| \leq \aleph_0$, and thus $|J| = \aleph_{j+1}$. $\square$

$\square$

Proof of Theorem 1.4: 2 $\Rightarrow$ 1 was Corollary 4.7, and 1 $\Rightarrow$ 3 is immediate. All that is left is to show that if 2 fails then so does 3.

So suppose T has a minimal type $p \in S(\mathrm{acl}(\emptyset))$ with an independent orbit, and say

$$p \perp^a f(p) \otimes f^2(p) \otimes \dots$$

where $f \in \mathrm{Aut}(\mathfrak{C})$. Then p cannot be strongly minimal, so by Buechler's dichotomy, p must be locally modular.

First suppose that p is nontrivial. If c is any realization of p, then each of the types in $\{f^i(p) : 1 \leq i < \omega\}$ is non-almost-orthogonal over c to a generic type $q \in S(\mathrm{acl}(\emptyset))$ of some weakly minimal, $\mathrm{acl}(\emptyset)$-definable group G (see [4] or [8]). By Fact 3.1, we may assume that G is abelian. We temporarily add constants to the language for the algebraic parameters used to define G so that G is definable over $\emptyset$, and let T' be this expanded language. There must be some finite power f^k of f which fixes these parameters, so without loss of generality f is still an automorphism in the language of T'. It follows that q also has an independent orbit, as witnessed by f again, and so $\overline{G}$ has an everywhere generic automorphism. By Theorem 3.13, the theory T' has infinitely many pairwise bi-embeddable, pairwise nonisomorphic models, and by Lemma 2.3, so does the original theory T.

Finally, suppose that p is trivial. Then the types in $\{f^i(p) : i < \omega\}$ are pairwise orthogonal (see [1]). By Theorem 4.8, we are done. $\square$

Remark 4.12. It seems that the Dushnik-Miller argument used in section 3 for weakly minimal groups could also be applied to weakly minimal theories in which there is a trivial type with an independent orbit, yielding a more uniform proof of Theorem 1.4 which avoids the construction in the proof of Theorem 4.8. There are some technical obstacles to doing this, however, so we have not included this argument in the present write-up.

References

[1] John B. Goode, “Some trivial considerations," *Journal of Symbolic Logic* **56** (1991), number 2, 624–631.

[2] Ben Dushnik and E. W. Miller, “Partially ordered sets," *American Journal of Mathematics* **63** (1941), number 3, 600–610.

[3] John Goodrick, “When are elementarily bi-embeddable structures isomorphic?," Ph. D. thesis, University of California, Berkeley, 2007.

[4] Ehud Hrushovski, “Almost orthogonal regular types," *Annals of Pure and Applied Logic* **45** (1939), 139–155.

[5] Thomas Jech, *Set Theory*, Academic Press, 1978.

[6] T. A. Nurmagambetov, "Characterization of ω-stable theories with a bounded number of dimensions," *Algebra i Logika* **28** (1989), number 5, 388–396.

[7] T. A. Nurmagambetov, "The mutual embeddability of models," from *Theory of Algebraic Structures* (in Russian), Karagand. Gos. Univ. 1985, pp. 109–115.

[8] Anand Pillay, *Geometric Stability Theory*, Oxford University Press, 1996.

[9] Saharon Shelah, *Classification Theory* (2nd edition), North-Holland (1990).